Restricted dissimilarity functions and penalty functions

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Abstract

In this work we introduce the definition of restricted dissimilarity functions and we link it with some other notions, such as metrics. In particular, we also show how restricted dissimilarity functions can be used to build penalty functions.

Keywords: Restricted dissimilarity functions, penalty functions, metrics.

1. Introduction

Given a set of inputs, penalty functions (see [4, 5, 7]) allow us to determine the aggregation function M_j that provides the output y_j which is the least dissimilar to the set of inputs $\{x_1, \cdots, x_p\}$. On the other hand, dissimilarity functions provide a way to measure how dissimilar two inputs are. This idea has led us to consider the use of restricted dissimilarity functions [2] to build penalty functions. To this end, we first investigate the conditions under which we can build convex or quasi-convex restricted dissimilarity functions, and then we analyze the relation of such functions with the so-called faithful dissimilarity functions (see [7]).

The structure of this work is as follows. In the next section we present some preliminaries. In Section 3 we present the concept of restricted dissimilarity functions and some related results. In Section 4 we focus on convex and quasi-convex restricted dissimilarity functions and their relation with metrics. In Section 5 we present some construction methods. In Sections 6 and 7 we consider faithful restricted dissimilarity functions and their relations with penalty functions. We finish with some conclusions and references.

2. Preliminaries

Definition 1 *A mapping $M : [0,1]^n \to [0,1]$ is an aggregation function if it is monotone non-decreasing in each of its components and satisfies $M(0,0,\cdots,0) = 0$ and $M(1,1,\cdots,1) = 1$.*

Definition 2 *An aggregation function M is called averaging or a mean if*

$$\min(x_1, \cdots, x_n) \leq M(x_1, \cdots, x_n) \leq \max(x_1, \cdots, x_n)$$

for all $x_1, \ldots, x_n \in [0,1]$.

Any averaging aggregation function is idempotent, and also the converse is true.

As stated in the Introduction, it is often necessary to measure the difference or disagreement between a set of inputs $(x_1, \cdots, x_p)$ and the corresponding output y. A possible way of performing such a measure is by means of so-called *penalty functions.* The idea is: given a penalty function, use it as a measure of dissimilarity by finding the aggregation function that minimizes the difference between inputs and output.

Definition 3 *A penalty function is a mapping $P : [0,1]^{n+1} \to \mathbb{R}^+ = [0,\infty]$ such that:*

1. *$P(\mathbf{x}, y) \geq 0$ for all $\mathbf{x} \in [0,1]^n, y \in [0,1]$;*
2. *$P(\mathbf{x}, y) = 0$ if and only if $x_i = y$ for all $i = 1, \cdots, n$;*
3. *$P(\mathbf{x}, y)$ is quasi-convex in y for any $\mathbf{x}$; that is, for each fixed $\mathbf{x} \in [0,1]^n$ the inequality*

$$P(\mathbf{x}, \lambda \cdot y_1 + (1-\lambda) \cdot y_2) \leq \max(P(\mathbf{x}, y_1), P(\mathbf{x}, y_2))$$

holds for any $\lambda \in [0,1]$ and any $y_1, y_2 \in [0,1]$.

Definition 4 *Let P be a penalty function. We assign the name penalty function based function (or function based on the penalty function P) to the mapping*

$$f(\mathbf{x}) = \arg\min_y P(\mathbf{x}, y),$$

if y is the only minimum and $y = \frac{a+b}{2}$ if the set of minimums is given by the interval $[a,b]$.

A penalty based aggregation function is always averaging. The following theorem states that the converse also holds.

Theorem 1 *[4] Any averaging aggregation function can be represented as a penalty based function in the sense of Definition 3.*

3. Restricted dissimilarity functions

In [2], the concept of *restricted dissimilarity function* was introduced as a tool to measure the dissimilarity between two given data. Moreover, different theorems of construction and characterization were considered in that work. In particular, restricted dissimilarity functions were also used to build distances between fuzzy sets in the sense of Liu [6].

Definition 5 *[2] A mapping $d_R : [0,1]^2 \to [0,1]$ is a restricted dissimilarity function if:*

1. *$d_R(x,y) = d_R(y,x)$ for every $x,y \in [0,1]$;*
2. *$d_R(x,y) = 1$ if and only if $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$; that is, if $\{x,y\} = \{0,1\}$;*
3. *$d_R(x,y) = 0$ if and only if $x = y$;*
4. *For any $x,y,z \in [0,1]$, if $x \le y \le z$, then $d_R(x,y) \le d_R(x,z)$ and $d_R(y,z) \le d_R(x,z)$.*

Notice that, contrary to the case of dissimilarity functions, restricted dissimilarity functions vanish if and only if both inputs are equal.

We will say that d_R is a strict restricted dissimilarity function if for any $x,y,z \in [0,1]$, if $x < y < z$, then $d_R(x,y) < d_R(x,z)$ and $d_R(y,z) < d_R(x,z)$.

Example 1 *The mapping $d_R(x,y) = |x-y|$ provides a simple example of a restricted dissimilarity function which is strict. On the other hand, as an example of a non-strict restricted dissimilarity function we can present the following. Take $c \in]0,1[$. Then*

$$d_R(x,y) = \begin{cases} 1 & \text{if } \{x,y\} = \{0,1\}\ ; \\ 0 & \text{if } x = y; \\ c & \text{otherwise.} \end{cases}$$

is a restricted dissimilarity function. Observe that this mapping is not even continuous.

Recall that a fuzzy negation is a non-increasing mapping $N : [0,1] \to [0,1]$ such that $N(0) = 1$ and $N(1) = 0$. a fuzzy negation is called strict if N is strictly decreasing. An involutive fuzzy negation is called a strong negation.

Theorem 2 *Let $d : [0,1]^2 \to [0,1]$ be a function. The following statements are equivalent.*

(i) *$N : [0,1] \to [0,1]$ is a strict negation and $d(x,y) = |N(x) - N(y)|$;*

(ii) *d is a restricted dissimilarity function, $d(x,y) = |d(x,1) - d(y,1)|$ and $d(1,x)$ is strictly monotone;*

(iii) *d is a restricted dissimilarity function and for all $x,y,z \in [0,1]$ with $x \ge y \ge z$, it holds that $d(x,y) + d(y,z) = d(x,z)$ and $d(1,x)$ is strictly monotone.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $d(x,y) = |N(x) - N(y)|$. Then, symmetry of d is obvious. Moreover, $d(x,y) = 0$ if and only if $N(x) = N(y)$, and since N is strict, this can happen if and only if $x = y$. On the other hand, $d(x,y) = 1$ if and only if $\{N(x), N(y)\} = \{0,1\}$, and once again from the strictness of N, this is equivalent to $\{x,y\} = \{0,1\}$. Finally, if $x \le y \le z$, then $d(x,y) = |N(x) - N(y)| = N(x) - N(y) \le N(x) - N(z) = |N(x) - N(z)|$. The case $d(y,z) \le d(x,z)$ is analogous. Observe that if $x < y < z$, then $d(x,y) < d(x,z)$ and $d(y,z) < d(x,z)$, due again to the strictness of d. Finally, $N(x) = |N(x) - N(1)| = d(x,1)$, so $d(x,y) = |d(x,1) - d(y,1)|$, as stated.

(ii) $\Rightarrow$ (iii) Take $x > y > z$. Then

$$\begin{aligned} d(x,y) + d(y,z) &= |N(x) - N(y)| + |N(y) - N(z)| \\ &= N(y) - N(x) + N(z) - N(y) = N(z) - N(x) \\ &= |N(x) - N(z)| = d(x,z)\ . \end{aligned}$$

The strict monotonicity of $d(1,x)$ follows from the symmetry of d and the fact that $d(x,y) = |d(1,x) - d(1,y)| = 0$ if and only if $x = y$.

(iii) $\Rightarrow$ (i) Define $N(x) = d(1,x)$. First of all, since $d(1,x)$ is strictly monotone, $d(1,1) = 0$ and $d(1,0) = 1$, it follows that N is a strict negation. Moreover, if $1 \ge y \ge z$, it follows that

$$d(1,y) + d(y,z) = d(1,z)$$

so $d(y,z) = d(1,z) - d(1,y)$, and since $y > z$, $d(y,z) = |d(1,y) - d(1,z)| = |N(y) - N(z)|$ $\square$

The proof of the following result is direct.

Proposition 1 *Let d_R be a restricted dissimilarity function. Then*

1. *$N(x) = d_R(1,x)$ for all $x \in [0,1]$ is a fuzzy negation;*
2. *$N(x) = d_R(1,x)$ is an involutive negation if and only if $d_R(1, d_R(1,x)) = x$ for all $x \in [0,1]$;*
3. *If $N(x) = d_R(1,x)$ and $d_R(x,y) = d_R(N(x), N(y))$ for all $x,y \in [0,1]$, then $N(x) = d_R(0, N(x))$ for all $x \in [0,1]$;*
4. *If $N(x) = d_R(1,x)$ and $d_R(x,y) = d_R(N(x), N(y))$ for all $x,y \in [0,1]$, then $d(1, d(1,x)) = x$ for all $x \in [0,1]$; that is, N is an involutive negation.*

Item 1) in Proposition 1 can be extended as follows.

Proposition 2 *Let d_R be a restricted dissimilarity function. Then, for any $y \in]0,1]$, the mapping*

$$N(x) = \frac{d_R(yx, y)}{d_R(0,y)}$$

is a negation.

Proof. N is well defined since we are taking $y \in]0,1]$ and if $y \neq 0$, then $d_R(0,y) \neq 0$. Moreover

$$0 \le yx \le y \text{ so } d_R(yx, y) \le d_R(0,y)$$

and $0 \le N(x) \le 1$. To conclude, observe that

$$N(0) = \frac{d_R(0,y)}{d_R(0,y)} = 1$$

and

$$N(1) = \frac{d_R(y,y)}{d_R(0,y)} = 0 \quad \Box$$

Regarding the relation with automorphisms, we can state the following.

Proposition 3 *Let d_R be a restricted dissimilarity function which is continuous and strict. Then, for each $y \neq 1$ the mapping*

$$\varphi(x) = \frac{d_R((1-y)x + y, y)}{d_R(1,y)}$$

is an automorphism on the unit interval.

Proof. It is an easy calculation. Notice that the denominator never vanishes $\Box$

4. Convex and quasi-convex restricted dissimilarity functions: Relation between restricted dissimilarity functions and metrics

Since we are going to relate restricted dissimilarity functions and penalty functions, convexity and quasi-convexity is a crucial property to be taken into account.

Theorem 3 *Let $d_R : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function. Then d_R is quasi-convex in one variable; that is, for all $x, y_1, y_2, \lambda \in [0,1]$*

$$d_R(x, \lambda \cdot y_1 + (1-\lambda) \cdot y_2) \leq \max(d_R(x,y_1), d_R(x,y_2))$$

Proof. Take $x, \lambda \in [0,1]$. We know that, for all $y_1, y_2 \in [0,1]$

$$\min(y_1,y_2) \leq \lambda \cdot y_1 + (1-\lambda) \cdot y_2 \leq \max(y_1,y_2)$$

There are three possibilities.
i) $x \leq \min(y_1,y_2)$. Then $d_R(x, \lambda \cdot y_1 + (1-\lambda) \cdot y_2) \leq d_R(x, \max(y_1,y_2)) \leq \max(d_R(x,y_1), d_R(x,y_2))$;
ii) $\min(y_1,y_2) \leq x \leq \max(y_1,y_2)$. In this situation two things can happen:
a) $\min(y_1,y_2) \leq x \leq \lambda \cdot y_1 + (1-\lambda) \cdot y_2 \leq \max(y_1,y_2)$, so $d_R(x, \lambda \cdot y_1 + (1-\lambda) \cdot y_2) \leq d_R(x, \max(y_1,y_2)) \leq \max(d_R(x,y_1), d_R(x,y_2))$, or
b) $\lambda \cdot y_1 + (1-\lambda) \cdot y_2 \leq x \leq \max(y_1,y_2)$, so $d_R(x, \lambda \cdot y_1 + (1-\lambda) \cdot y_2) \leq d_R(x, \min(y_1,y_2)) \leq \max(d_R(x,y_1), d_R(x,y_2))$;
iii) $\max(y_1,y_2) \leq x$, can be treated as item i) $\Box$

Corollary 1 *Let $d_R : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function. If $\lambda_0, \lambda_1 \in [0,1]$, then*

$$d_R(\lambda_0 \cdot x_1 + (1-\lambda_0) \cdot x_2, \lambda_1 \cdot y_1 + (1-\lambda_1) \cdot y_2) \leq \\ \max(d_R(x_1,y_1), d_R(x_1,y_2), d_R(x_2,y_1), d_R(x_2,y_2))$$

for all $x_1, x_2, y_1, y_2 \in [0,1]$.

Proof. From the symmetry of d_R we have $d_R(\lambda_0 \cdot x_1 + (1-\lambda_0) \cdot x_2, \lambda_1 \cdot y_1 + (1-\lambda_1) \cdot y_2) \leq \max(d_R(\lambda_0 \cdot x_1 + (1-\lambda_0) \cdot x_2, y_1), d_R(\lambda_0 \cdot x_1 + (1-\lambda_0) \cdot x_2, y_2))$, and the result follows from the quasi-convexity in each variable. $\Box$

For the remainder of this paper, whenever we say that d_R is convex or quasi-convex (concave or quasi-concave) we mean that it is so in both variables. Otherwise, we will state it explicitly.

Theorem 4 *Let $d_R : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function which is concave in one coordinate. Then d_R is a metric on $[0,1]$.*

Proof. We have to check the triangle inequality of d_R only, as the other properties of metrics are trivially fulfilled by d_R. The only non-trivial case to be checked is when $0 \leq x < z < y \leq 1$. Then the concavity in one coordinate ensures for each $\lambda \in [0,1]$ that

$$\begin{aligned} d_R(x, \lambda x + (1-\lambda)y) &\geq \lambda d_R(x,x) + (1-\lambda) d_R(x,y) \\ &= (1-\lambda) d_R(x,y) \ . \end{aligned}$$

Similarly,

$$d_R(\lambda x + (1-\lambda)y, y) \geq \lambda d_R(x,y) \ .$$

Thus

$$\begin{aligned} d_R(x,y) &= \tfrac{y-z}{y-x} d_R(x,y) + \tfrac{z-x}{y-x} d_R(x,y) \\ &\leq d(x,z) + d(y,z) \ , \end{aligned}$$

proving that d_R is a metric $\Box$

Remark

(i) d_R satisfying Theorem 4 is necessarily strict.
(ii) $d_R(x,y) = (x-y)^2$ is a strict dissimilarity restricted function which is not a metric (note that it is nor concave in one coordinate).

5. Some construction methods of restricted dissimilarity functions

Proposition 4 *Let $\varphi : [0,1] \to [0,1]$ be an automorphism and $d_{R_1} : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function. Then $d_R(x,y) = d_{R_1}(\varphi(x), \varphi(y))$ is a restricted dissimilarity function. Moreover, if $d_{R_1}(x,1)$ is strictly monotone, $d_R(x,y) = d_{R_1}(x,y)$ for all $x,y \in [0,1]$ if and only if $\varphi(x) = x$ for all $x \in [0,1]$.*

Proof. The fact that d_R as defined in the Proposition as a restricted dissimilarity function is clear. To see the last point, assume that $d_{R_1}(x,y) = d_R(x,y)$. Then, in particular, $d_{R_1}(1,y) = d_R(1, \varphi(y))$, and the result follows from the monotonicity of $d_{R_1}(1,x)$ $\Box$

Proposition 5 *If φ_1, φ_2 are two automorphisms of the unit interval, then*

$$d(x,y) = \varphi_1(|\varphi_2(x) - \varphi_2(y)|)$$

is a strict dissimilarity function.

Proposition 6 *Let φ be an automorphism and $K : [0,1] \to [0,1]$ a mapping such that:*

1. *$K(x) = 0$ if and only if $x = 0$;*
2. *$K(x) = 1$ if and only if $x = 1$;*
3. *K is increasing.*

Then

$$d_R(x,y) = K(|\varphi(x) - \varphi(y)|)$$

is a restricted dissimilarity function.

Proof. Direct □

Corollary 2 *Let d_R be a continuous and strict restricted dissimilarity function. Then the mapping*

$$d'_R(x,y) = d_R(0, |d_R(x,1) - d_R(y,1)|)$$

is also a continuous and strict restricted dissimilarity function.

Proposition 7 *Let d_R and d'_R be defined as in the previous corollary. Then, $d_R(x,y) = d'_R(x,y)$ for every $x, y \in [0,1]$ if and only if*

$$d_R(x,y) = |d_R(x,1) - d_R(y,1)|$$

holds.

Proof. Necessity is quite easy. Just observe that, if $d_R(x,y) = d_R(0, |d_R(x,1) - d_R(y,1)|)$, then, for all $x \in [0,1]$, $d_R(x,1) = d_R(0, d_R(x,1))$. Since $\varphi(x) = d_R(0,x)$ is an automorphism on the unit interval and d_R is continuous and strict, we see that $\varphi(x) = x$ for all $x \in [0,1]$. The converse is straightforward □

Taking $y = 1$ we see that

$$d'_R(x,1) = d_R(0, d_R(x,1))$$

Other construction methods of restricted dissimilarity functions can be found in [2].

5.1. Construction of convex or quasi-convex restricted dissimilarity functions

A practical method to build penalty functions is by means of convex functions. As we intend to use restricted dissimilarity functions to get penalty functions, it is natural to consider methods for constructing convex or quasi-convex restricted dissimilarity functions. Regarding convexity, the last condition in the definition of restricted dissimilarity functions can be seen as a weak form of convexity.

Proposition 8 *Let $d_R : [0,1]^2 \to [0,1]$ be a symmetric quasi-convex function such that*

1. *$d_R(x,y) = 0$ if and only if $x = y$;*
2. *$d_R(x,y) = 1$ if and only if $\{x,y\} = \{0,1\}$.*

Then d_R is a restricted dissimilarity function.

Proof. Take $x \leq y \leq z$. Then there exists $\lambda \in [0,1]$ such that $y = \lambda x + (1-\lambda)z$. From the symmetry of d_R,

$$\begin{aligned} d_R(y,z) &= d_R(\lambda x + (1-\lambda)z, z) \\ &\leq \max(d_R(x,z), d_R(z,z)) = d_R(x,z) \ . \end{aligned}$$

Moreover, we also have that

$$\begin{aligned} d_R(x,y) &= d_R(x, \lambda x + (1-\lambda)z) \\ &\leq \max(d_R(x,x), d_R(x,z)) = d_R(x,z) \Box \end{aligned}$$

Corollary 3 *Let d_R be a symmetric convex function such that*

1. *$d_R(x,y) = 0$ if and only if $x = y$;*
2. *$d_R(x,y) = 1$ if and only if $\{x,y\} = \{0,1\}$.*

Then, d_R is a strict restricted dissimilarity function.

Proof. Since any convex function is also quasi-convex, d_R is a restricted dissimilarity function. To see strictness, take $x < y < z$. Then there exists $\lambda \in]0,1[$ such that $y = \lambda x + (1-\lambda)z$. So

$$\begin{aligned} d_R(y,z) &= d_R(\lambda x + (1-\lambda)z, \lambda z + (1-\lambda)z) \\ &\leq \lambda d_R(x,z) + (1-\lambda) d_R(z,z) < d_R(x,z) \end{aligned}$$

where the last inequality follows from $d_R(x,z) > 0$, $x \neq z$ and $\lambda \in]0,1[$ □

Corollary 4 *Let d_R be a symmetric and strictly convex function such that*

1. *$d_R(x,x) = 0$ for all $x \in [0,1]$;*
2. *$d_R(x,y) = 1$ if and only if $\{x,y\} = \{0,1\}$.*

Then, d_R is a strict restricted dissimilarity function.

Proof. If $y > x$, we have that $x < \frac{x+y}{2} < y$, and

$$0 \leq d_R(\frac{x+y}{2}, y) < \frac{1}{2} d_R(x,y) + \frac{1}{2} d_R(y,y) \leq d_R(x,y)$$

so $d_R(x,y) > 0$ as long as $x \neq y$. The result follows from Corollary 3 □

6. Faithful restricted dissimilarity functions

In [4, 5, 7] the following concept is introduced to build penalty functions.

Definition 6 *The function $p : X^2 \to \mathcal{R}_+$ is called a faithful penalty function if it satisfies $p(x,y) = 0$ if and only if $x = y$ and it can be represented as $p(t,s) = \mathbf{K}(h(t), h(s))$, where $h : X \to \mathcal{R}$ is some continuous monotone function (scaling function) and $\mathbf{K} : \mathcal{R}^2 \to \mathcal{R}_+$ is convex.*

Here $X = [a,b]^n \subseteq \bar{\mathcal{R}} = [-\infty, +\infty]$. In this paper we will take $p : [0,1]^2 \to [0,1]$, $h : [0,1] \to [0,1]$ and $\mathbf{K} : [0,1]^2 \to [0,1]$.

Proposition 9 *In the setting of Proposition 4, if d_{R_1} is convex, then the restricted dissimilarity function d_R is a faithful penalty function.*

Proof. Direct □

A special class of faithful penalty functions are the faithful dissimilarity functions defined in [7]. Faithful dissimilarity functions p are expressed as

$$p(x, y) = K(h(x) - h(y))$$

where $K : \mathcal{R} \to \mathcal{R}$ is convex with the unique minimum $K(0) = 0$ and h is a strictly monotone continuous function $h : X \to \mathcal{R}$. Observe that these functions are continuous. In this work, we take $K : [-1, 1] \to [0, 1]$.

We will use the term *faithful restricted dissimilarity functions* for faithful dissimilarity functions which are also restricted dissimilarity functions.

Proposition 10 *In the setting of Proposition 6, if K is convex, then d_R is a faithful restricted dissimilarity function.*

Proof. Direct □

Example 2 *1. If $K(x) = x^2$, then $d_R(x, y) = (h(x) - h(y))^2$,*
2. If $K(x) = |x|$, then $d_R(x) = |h(x) - h(y)|$.

Next we characterize a particular case of faithful restricted dissimilarity functions.

Lemma 1 *Let d_R be a faithful restricted dissimilarity function such that*

$$d_R(x, y) = K(h(x) - h(y))$$

with $K : [-1, 1] \to [0, 1]$ convex with a unique minimum at $K(0) = 0$ and $h : [0, 1] \to [0, 1]$ continuous and strictly monotone. Then there exists a convex $K_0 : [-1, 1] \to [0, 1]$ with a unique minimum at $K(0) = 0$ such that if $0 < x < y$, then $K_0(x) < K_0(y)$ in such a way that

$$d_R(x, y) = K_0(h(x) - h(y)) \ .$$

Proof. For the mapping $K : [-1, 1] \to [0, 1]$ in the statement it holds that for $0 < x < y$, $K(x) < K(y)$, there is nothing to prove. So assume that this is not the case and h is increasing (the decreasing case is analogous). The mapping

$$F : [0, h(1) - h(0)] \to [0, 1] \text{ given by}$$
$$x \to K(h(x) - h(0))$$

is a bijection since:
a) F is continuous, since K is convex and h is continuous;
b) $F(0) = K(h(0) - h(0)) = K(0) = 0$;
c) $F(1) = K(h(1) - h(0)) = d_R(1, 0) = 1$;
d) From continuity and items b) and c) F is surjective;
e) If $F(x) = F(y)$, then $K(h(x) - h(0)) = K(h(y) - h(0))$; that is;

$$x = d_R(x, 0) = d_R(y, 0) = y \ .$$

So injectivity follows.

Consider the function:

$$r : [0, 1] \to [0, h(1) - h(0)] \text{ given by}$$
$$t \to (h(1) - h(0))t$$

where r is increasing and $K \circ r$ is convex. Moreover, r is continuous and bijective, so $K \circ r$ is also continuous and bijective. Defining $K_0 = K \circ r$ and taking into account that K must be symmetric, the proof is complete □

Theorem 5 *Take $d_R : [0, 1]^2 \to [0, 1]$. Then the following items are equivalent.*

i) d_R is a faithful restricted dissimilarity function that satisfies $d_R(x, y) = d_R(N(x), N(y))$ with $N(x) = d_R(1, x)$.
ii) There exists a concave automorphism φ on the unit interval such that

$$d_R(x, y) = \varphi^{-1}(|\varphi(x) - \varphi(y)|)$$

Proof. i) ⇒ ii) As d_R is a faithful restricted dissimilarity function, there is a convex $K : [-1, 1] \to [0, 1]$ such that $K(x) = 0$ if and only if $x = 0$ (unique minimum at $x = 0$) and there exists $h : [0, 1] \to [0, 1]$ continuous and strictly monotone such that

$$d_R(x, y) = K(h(x) - h(y))$$

On the other hand, $N(x) = d_R(1, x)$ is a strong negation. So there exists an automorphism $\varphi : [0, 1] \to [0, 1]$ such that

$$N(x) = d_R(1, x) = \varphi^{-1}(1 - \varphi(x))$$

and

$$d_R(1, x) = K(h(1) - h(x)) \ .$$

We also know that $N(x) = d_R(1, x) = d_R(N(1), N(x)) = d_R(0, N(x))$. As N is a strong negation, $x = d_R(0, x)$ for all $x \in [0, 1]$. Whence

$$x = d_R(0, x) = d_R(x, 0) = K(h(x) - h(0)) \ .$$

Taking $x = 1$ we have $1 = K(h(1) - h(0))$

On the other hand, from Lemma 1 if $0 < x < y$, then $K(x) \neq K(y)$.

If K is strictly convex, $h(1) - h(0) = 1$. So:
a) either h is an automorphism or
b) h is a strict negation.

a) If h is an automorphism, then $x = d_R(x, 0) = K(h(x) - h(0)) = K(h(x))$. As K is injective in $[0, 1]$ it holds that $K^{-1}(x) = h(x)$ is an automorphism and as K is convex, K^{-1} is concave.

Take $\varphi(x) = K^{-1}(x)$. Then $h(x) = \varphi^{-1}(x)$ and

$$d_R(x, y) = \begin{cases} \varphi^{-1}(\varphi(x) - \varphi(y)) & \text{if } x \geq y \ ; \\ \varphi^{-1}(\varphi(y) - \varphi(x)) & \text{otherwise.} \end{cases}$$

That is; $d_R(x, y) = \varphi^{-1}(|\varphi(x) - \varphi(y)|)$.

b) can be proved analogously □

7. Faithful restricted dissimilarity functions and penalty based functions

Theorem 6 *Let $M : [0,1]^n \to [0,1]$ be an aggregation function and $d_R : [0,1]^2 \to [0,1]$ a faithful restricted dissimilarity function. Then*

$$P : [0,1]^{n+1} \to [0,1] \text{ given by}$$
$$P(\mathbf{x}, y) = \overset{n}{\underset{i=1}{M}} d_R(x_i, y)$$

is a penalty function.

Proof. (1) $P(\mathbf{x}, y) \geq 0$ for all $\mathbf{x} \in [0,1]^n$ and $y \in [0,1]$. (2) If $y = x_i$ for all $i = 1, \cdots, n$, then $d_R(x_i, y) = 0$, then $P(\mathbf{x}, y) = 0$. (3) As d_R is a faithful restricted dissimilarity function, we can write it as $d_R(x_i, y) = K(h(x_i) - h(y))$ with $K : [-1,1] \to [0,1]$ a convex function with a unique minimum at 0 ($K(0) = 0$) and $h : [0,1] \to [0,1]$ continuous and strictly monotone. So d_R is quasiconvex.

If h is increasing (the decreasing case is analogous)

$$\begin{aligned}
& P(\mathbf{x}, \lambda y_1 + (1-\lambda) y_2) \\
&= \overset{n}{\underset{i=1}{M}} d_R(x_i, \lambda y_1 + (1-\lambda) y_2) \\
&= \overset{n}{\underset{i=1}{M}} K(h(x_i) - h(\lambda y_1 + (1-\lambda) y_2)) \\
&\leq \begin{cases} MK(h(x_i) - h(\max(y_1, y_2))) \\ \text{ if } h(x_i) - h(\lambda y_1 + (1-\lambda) y_2) \leq 0; \\ MK(h(x_i) - h(\min(y_1, y_2))) \\ \text{ in other case.} \end{cases}
\end{aligned}$$

An in any case, we have that this is less than or equal to $\max(\overset{n}{\underset{i=1}{M}} K(h(x_i) - h(y_1)), \overset{n}{\underset{i=1}{M}} K(h(x_i) - h(y_2)))$ □

Corollary 5 *In the setting of Theorem 6, if $M : [0,1]^n \to [0,1]$ is an aggregation function such that $M(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$ and $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$ holds, then*

1. *$P(\mathbf{x}, y) = 0$ if and only if $\{x_i, y\} = \{0, 1\}$ for all $i = 1, \cdots, n$;*
2. *$P(\mathbf{x}, y) = 1$ if and only if $x_i = y$ for all $i = 1, \cdots, n$;*
3. *If $\mathbf{x} = (1, \cdots, 1)$, then $P(\mathbf{x}, y) = \overset{n}{\underset{i=1}{M}} n(y)$ with $n(y) = d_R(1, y)$;*
4. *$P(\mathbf{x}, 1) = \overset{n}{\underset{i=1}{M}} n(x_i)$ with $n(x_i) = d_R(1, x_i)$ for all $i = 1, \cdots, n$;*
5. *If φ is a concave automorphism, then*
$$P(\mathbf{x}, y) = \overset{n}{\underset{i=1}{M}} \varphi^{-1}(|\varphi(x_i) - \varphi(y)|)$$
is a penalty function.

Proof. Direct □

Corollary 6 *In the setting of Theorem 6, if M is idempotent, then if $\mathbf{x} = (1, \cdots, 1)$, $P(\mathbf{x}, y) = n(y)$ with $n(y) = d_R(1, y)$.*

Proof. Direct □

Proposition 11 *Let $M : [0,1]^n \to [0,1]$ be an idempotent aggregation function and $d_R : [0,1]^2 \to [0,1]$ a faithful restricted dissimilarity function. Then the penalty based function*

$$f(\mathbf{x}) = \arg\min_y P(\mathbf{x}, y) = \arg\min_y \overset{n}{\underset{i=1}{M}} d_R(x_i, y)$$

is an idempotent function.

Proof. Direct □

8. Conclusions

In this work we have recalled the concept of restricted dissimilarity functions and have related it to penalty functions via faithful restricted dissimilarity functions. We have also shown some connections between restricted dissimilarity functions and metrics.

In the future we intend to develop further the theoretical aspects of this work, specially analyzing what kinds of averaging aggregation functions can be generated by the use of penalty functions built from restricted dissimilarity functions.

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